Sydney Grammar School
Mathematics Department
Trial Examinations 2006

FORM VI

MATHEMATICS

Examination date

Tuesday 1st August 2006

Time allowed

Three hours (plus 5 minutes reading time)

Instructions

All ten questions may be attempted.
All ten questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the ten questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 10 per boy. A total of 1250 booklets should be sufficient.
Candidature: 105 boys.

Examiner

REN

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $x^2 - 3x - 10 = 0$. $\boxed{1}$

(b) Convert $\frac{5\pi}{9}$ radians to degrees. $\boxed{1}$

(c) Find a primitive of $x^5 - 1$. $\boxed{1}$

(d) Use your calculator to find $\dfrac{15{\cdot}7}{\sqrt{1{\cdot}6 + 2{\cdot}9}}$, giving your answer correct to one decimal place. $\boxed{2}$

(e) Write $\dfrac{7}{\sqrt{5} - 2}$ with a rational denominator. $\boxed{2}$

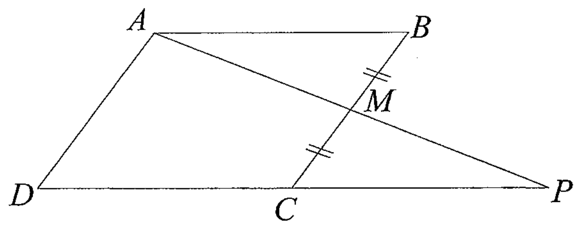
(f) Solve $|x - 3| = 1$. $\boxed{1}$

(g) Sketch the graph of $x^2 + y^2 = 9$, showing all x and y-intercepts. $\boxed{2}$

(h) A man's weekly income is \$1650. If he receives a 4% wage increase, find his new weekly income. $\boxed{2}$

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a)



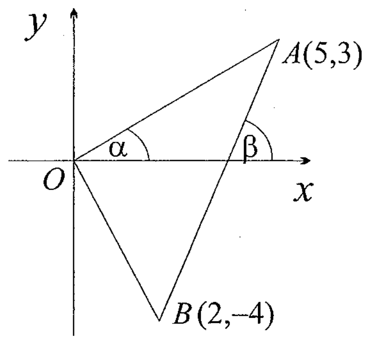
The diagram above shows the parallelogram $ABCD$ with M the midpoint of BC. The intervals AM and DC are produced to meet at P.

(i) Prove that $\triangle ABM \equiv \triangle PCM$. $\boxed{2}$

(ii) Hence prove that $ABPC$ is a parallelogram. $\boxed{2}$

Exam continues next page ...

(b)



The diagram above shows $\triangle AOB$ with A and B the points $(5,3)$ and $(2,-4)$ respectively. The angle of inclination of OA is α and the angle of inclination of AB is β.

(i) Write down the gradients of OA and AB. [2]

(ii) Hence find α and β, both correct to the nearest degree. [2]

(iii) Find the length of OA. [1]

(iv) Find the length of AB. [1]

(v) Find the area of $\triangle AOB$. Give your answer correct to two significant figures. [2]

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following with respect to x:

(i) $\log_e(4x+3)$ [1]

(ii) $x\sin x$ [1]

(b) (i) Find $\int e^{6x}\,dx$. [1]

(ii) Evaluate $\int_1^9 \sqrt{x}\,dx$. [2]

(c) The equation of a parabola is given by $(x-1)^2 = 8y$.

(i) Write down the coordinates of the vertex. [1]

(ii) Write down the focal length. [1]

(iii) Sketch the graph of the parabola clearly showing the focus and directrix. [3]

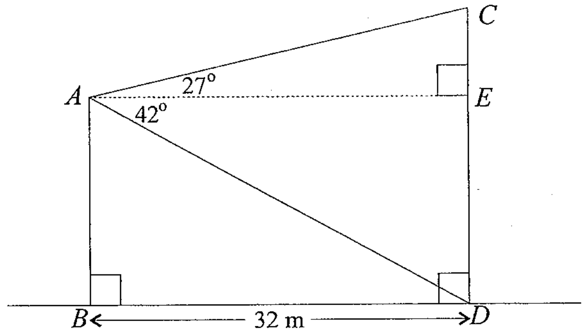
(d) Solve the equation $\tan x = \sqrt{3}$, for $0 \le x \le 2\pi$. [2]

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = x^2 - 3x$ at the point $(1,-2)$. [3]

(b)



In the diagram above, two vertical buildings AB and CD are 32 metres apart. From A, the angle of elevation of C is 27° and the angle of depression to D is 42°.

(i) Find the height of building AB, correct to two significant figures. [2]

(ii) Find the height of building CD, correct to two significant figures. [1]

(c) Find all solutions of the equation $x^4 - 7x^2 + 12 = 0$. [3]

(d) The table below shows the values of the function $f(x)$ for five values of x: [3]

x	4	4·5	5	5·5	6
$f(x)$	1·3	2·9	0·7	−0·2	−1·1

Use Simpson's rule with these five function values to find an estimate for $\int_4^6 f(x)\,dx$. Give your answer correct to one decimal place.

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Write down the domain and range of the function $y = 4\sin x$. [2]

(b) Consider the quadratic equation $x^2 - kx + (k+3) = 0$.

(i) Find the discriminant and write it in simplest form. [1]

(ii) For what values of k does the equation have no real roots? [2]

(iii) If the product of the roots is equal to three times the sum of the roots, find the value of k. [2]

Exam continues next page ...

(c) In order to study the history of the Earth's climate, a team of scientists drilled an "ice core" in the Antarctic ice sheet. They drilled 5 metres on the first day, a further 7 metres on the second day, a further 9 metres on the third day and so on.

(i) Find how many metres they drilled on the 40th day. $\boxed{1}$

(ii) Find how deep they had drilled after 40 days. $\boxed{1}$

(iii) Find how many days it took to drill to a depth of 480 metres. $\boxed{3}$

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The equation of a curve is given by $y = e^x - x$. Find the stationary point on the curve and determine its nature. $\boxed{3}$

(b) On a certain island, the population P of rabbits is increasing so that after time t weeks the value of P is given by $P = Ae^{kt}$, where A and k are constants.
When the population was first measured there were 200 rabbits on the island, and 5 weeks later there were 750 rabbits.

(i) Write down the value of A. $\boxed{1}$

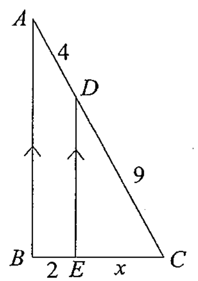
(ii) Find the exact value of k. $\boxed{1}$

(iii) Find the number of rabbits on the island after 12 weeks. (Give your answer correct to three significant figures.) $\boxed{1}$

(iv) Find how long it will take for the rabbit population to reach 100 000. (Give your answer correct to the nearest week.) $\boxed{2}$

(v) Find the rate at which the rabbit population is increasing after 4 weeks. (Give your answer correct to the nearest whole number.) $\boxed{2}$

(c) $\boxed{2}$



The diagram above shows $\triangle ABC$ with points D and E on AC and BC respectively so that $DE \parallel AB$. $AD = 4, DC = 9, BE = 2$ and $EC = x$.
Find the value of x, giving reasons.

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) A particle P is moving in a straight line so that its velocity v metres per second after t seconds is given by $v = 12 - 4t$. Initially P is 3 metres to the right of the origin O.

(i) Find the initial velocity and acceleration of P. $\boxed{2}$

(ii) If the displacement of P from O is x metres, find an expression for x in terms of t. $\boxed{2}$

(iii) Find when and where P is stationary. $\boxed{2}$

(iv) Sketch the graph of $v = 12 - 4t$, for $0 \le t \le 5$. $\boxed{1}$

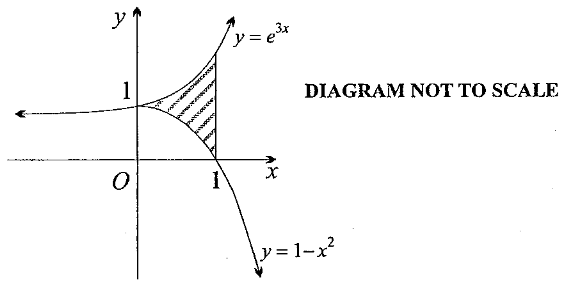
(v) Hence, or otherwise, find the total distance travelled by P during the first five seconds. $\boxed{2}$

(b) Solve the equation $\log_2 x + 1 = \log_2 \sqrt{x}$. $\boxed{3}$

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $\ln(\cos x)$ and express your answer in simplest form. $\boxed{2}$

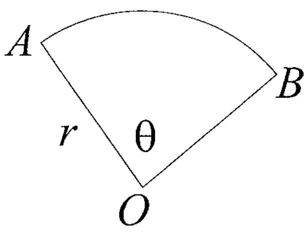
(b) $\boxed{3}$



The diagram above shows the region enclosed between the two curves $y = e^{3x}$ and $y = 1 - x^2$ and the line $x = 1$. Find the area of this region.

Exam continues next page ...

(c)



The diagram above shows a sector AOB, with radius r cm and with $\angle AOB = \theta$ radians. The area of the sector AOB is $40\,\text{cm}^2$.

(i) If P cm is the perimeter of sector AOB, show that $P = 2r + \frac{80}{r}$. $\boxed{1}$

(ii) Find the value of r and of θ for which P is a minimum. Justify your answer. $\boxed{3}$

(iii) Suppose now that, instead of $40\,\text{cm}^2$, the area of sector AOB is $x\,\text{cm}^2$. Find the value of θ for which P is a minimum. $\boxed{2}$

(iv) What general conclusion can you draw from part (iii)? $\boxed{1}$

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Sketch the graph of $y = \cos 2x$, for $0 \le x \le 2\pi$. (Your diagram should take up at least a quarter of a page.) $\boxed{2}$

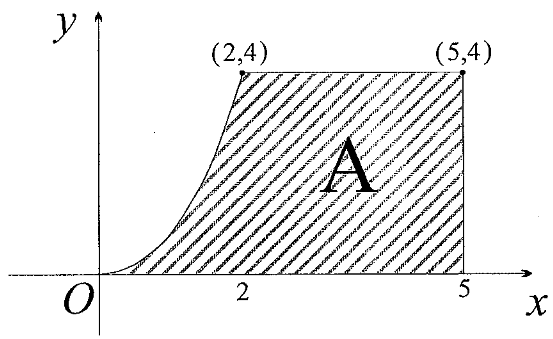
(ii) Copy and complete the following table of values for $y = \frac{1}{4}x$. (Give the values correct to one decimal place where necessary.) $\boxed{1}$

x	0	π	2π
y			

(iii) Use the table to sketch the graph of $y = \frac{1}{4}x$ on the same diagram as part(i). $\boxed{1}$

(iv) Hence determine the number of solutions of the equation $4\cos 2x - x = 0$. (Note: You do not have to solve this equation.) $\boxed{2}$

(b)



A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x^2, & \text{for } 0 \le x \le 2, \\ 4, & \text{for } 2 < x \le 5. \end{cases}$$

In the diagram above the shaded region A is bounded by the graph of $f(x)$, the x-axis and the line $x = 5$.

(i) Find the volume of the solid formed when the shaded region A is rotated about the x-axis. $\boxed{3}$

(ii) If the shaded region A is now rotated about the y-axis, find the volume of the solid formed. $\boxed{3}$

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Emily borrows \$80 000 in order to buy a home unit. The interest rate is 6% per annum reducible and the loan is to be repaid in equal monthly repayments over 20 years with the interest calculated monthly.
Let $\$A_n$ be the amount owing after the nth repayment.

(α) Write down expressions for $\$A_1$ and $\$A_2$, the amounts owing after the first and second repayments respectively. **1**

(β) Show that the amount of each monthly repayment is \$573·14 (correct to the nearest cent). **2**

(ii) After $2\frac{1}{2}$ years (i.e. 30 repayments) the interest rate rises to $7\frac{1}{2}$%. Find the new monthly repayment, correct to the nearest cent. (Assume that the period of the loan is still 20 years.) **3**

(b) (i) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$. **2**

(ii) Show that $\dfrac{d}{dx}(\sec x + \tan x) = \sec x(\sec x + \tan x)$. **1**

(iii) Hence, or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x\,dx$. **3**

END OF EXAMINATION

FORM VI MATHEMATICS TRIAL SOLUTIONS

Q1

(a) $(x-5)(x+2) = 0$
$x = 5$ or -2

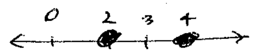
(b) $\frac{5\pi}{9} \times \frac{180}{\pi} = 100°$

(c) $\int (x^5 - 1)\,dx = \frac{x^6}{6} - x + c$

(d) 7·4 (1 dec. place)

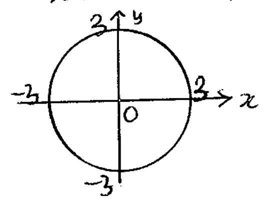
(e) $\frac{7}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{7(\sqrt{5}+2)}{5-4}$
$= 7(\sqrt{5}+2)$

(f)



$x = 2$ or 4

(g)



(h) New income $= 1.04 \times 1650$
$= \$1716$

(b) (i) grad $OA = \frac{3}{5}$
grad $AB = \frac{3--4}{5-2}$
$= \frac{7}{3}$

(ii) $\tan\alpha = \frac{3}{5}$
$\alpha \doteqdot 31°$
$\tan\beta = \frac{7}{3}$
$\beta \doteqdot 67°$

(iii) $OA = \sqrt{5^2 + 3^2}$
$= \sqrt{34}$

(iv) $AB = \sqrt{(5-2)^2 + (3--4)^2}$
$= \sqrt{3^2 + 7^2}$
$= \sqrt{58}$

(v) Area $\triangle AOB = \frac{1}{2} \times \sqrt{58} \times \sqrt{34} \times \sin 36°$
$\doteqdot 13$ units^2

Q2

(a) (i) In $\triangle$'s ABM, PCM
1. BM = MC (given)
2. ∠AMB = ∠PMC (vert. opp. ∠'s)
3. ∠ABM = ∠MCP (Alt ∠'s AB // CP)

$\triangle ABM \equiv \triangle PCM$ (ASA)

(ii) AM = MP (matching sides)
ABPC is a parallelogram (diag. BC, AP bisect each other at M)

Q3

(a) (i) $\frac{4}{4x+3}$

(ii) $x \times \cos x + 1 \times \sin x$
$= x\cos x + \sin x$

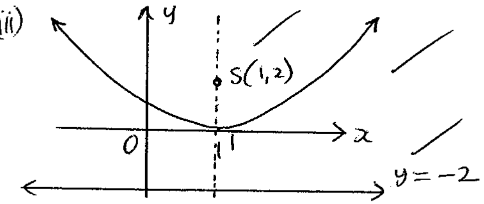
(b) (i) $\frac{1}{6}e^{6x} + c$

(ii) $\frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^9$
$= \frac{2}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$
$= \frac{52}{3}$

(c) (i) Vertex $(1, 0)$

(ii) Focal length $= 2$

(iii)



(d) related angle $= \frac{\pi}{3}$
$x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$

(c) let $a = x^2$
$a^2 - 7a + 12 = 0$
$(a-4)(a-3) = 0$
$a = 4$ or 3
$x^2 = 4$ or $x^2 = 3$
$x = 2, -2, \sqrt{3}$ or $-\sqrt{3}$.

(d) $\int_4^6 f(x)\,dx \doteqdot \int_4^5 f(x)\,dx + \int_5^6 dx$

$\doteqdot \frac{1}{6}\left[f(4) + 4\times f(4.5) + f(5)\right] + \frac{1}{6}\left[f(5) + 4\times f(5.5) + f(6)\right]$

$= \frac{1}{6}\left[1.3 + 4\times 2.9 + 2\times 0.7 + 4\times(-0.2) - 1.1\right]$

$\doteqdot 2.1$

Q4

(a) $\frac{dy}{dx} = 2x - 3$
At $(1, -2)$, grad $= 2\times 1 - 3$
$= -1$

Eqn of tangent : $y + 2 = -1(x - 1)$
$y = -x - 1$

(b) (i) $\tan 42^\circ = \frac{AB}{32}$
$AB = 32 \times \tan 42^\circ$
$\doteqdot 29$ m

(ii) $\tan 27^\circ = \frac{CE}{32}$
$CE = 32 \times \tan 27^\circ$

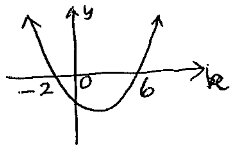
$CD = CE + AB$
$\doteqdot 45$ m

Q5

(a) Domain: $x \in \mathbb{R}$
Range: $-4 \le y \le 4$

(b) (i) $\Delta = (-k)^2 - 4\times 1\times(k+3)$
$= k^2 - 4k - 12$

(ii) $k^2 - 4k - 12 < 0$
$(k-6)(k+2) < 0$



$-2 < k < 6$

(iii) $3k = k + 3$
$2k = 3$
$k = 1\frac{1}{2}$

(c) (i) $5 + 7 + 9 + 11 + \cdots$
(AP, $a = 5$, $d = 2$)
$T_{40} = 5 + 39\times 2$
$= 83$ metres.

(ii) $S_{40} = \frac{40}{2}(5 + 83)$
$= 1760$ metres.

(iii) $480 = \frac{n}{2}\left[10 + 2(n-1)\right]$
$= n(5 + n - 1)$
$n^2 + 4n - 480 = 0$
$(n + 24)(n - 20) = 0$
$n = 20$
i.e. 20 days.

Q6

(a) $y' = e^x - 1$
At stat pt. $e^x - 1 = 0$
$e^x = 1$
$x = 0$

when $x = 0$, $y = e^0 - 0 = 1$
$y' = 0$ at $(0, 1)$

$y'' = e^x$
At $(0,1)$, $y'' = e^0 = 1$
Since $y'' > 0$, $(0,1)$ is a rel. min.

(b) (i) $A = 200$

(ii) $750 = 200e^{5k}$
$k = \frac{1}{5}\ln\frac{15}{4}$

(iii) when $t = 12$, $P = 200e^{\frac{12}{5}\ln\frac{15}{4}}$
$\doteqdot 4770$ rabbits

(iv) $100\,000 = 200e^{kt}$
$t = \frac{\ln 500}{k}$
$\doteqdot 24$ weeks.

(v) $\frac{dP}{dt} = Ake^{kt}$
when $t = 4$, $\frac{dP}{dt} = 200ke^{4k}$
$\doteqdot 152$ rabbits/week

(c) $\frac{EC}{EB} = \frac{CD}{DA}$ (Interval $\parallel$ to one side of triangle divides other two sides in proportion)

$\frac{x}{2} = \frac{9}{4}$
$x = 4\frac{1}{2}$

Q7

(a) (i) When $t=0$, $v = 12 - 4\times 0$
$= 12$ m/s

$\frac{dv}{dt} = -4$ m/s

(ii) $x = 12t - 2t^2 + c$

$t=0,\ x=3,\quad 3 = 0 - 0 + c$
$c = 3$

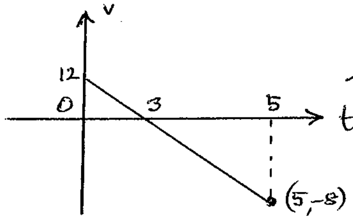
$x = 12t - 2t^2 + 3$

(iii) Stationary when $12 - 4t = 0$
$t = 3$ seconds

when $t=3$, $x = 12\times 3 - 2\times 3^2 + 3$
$= 21$

Stationary when $x = 21$ m

(iv)



(v) Distance $= \frac{12\times 3}{2} + \frac{2\times 8}{2}$
$= 26$ m.

(b) $\log_2 x + 1 = \log_2 \sqrt{x}$

$\log_2 x + 1 = \frac{1}{2}\log x$

$\frac{1}{2}\log_2 x = -1$

$\log_2 x = -2$

$x = 2^{-2}$

$x = \frac{1}{4}$.

Q8

(a) $\frac{d}{dx}(\ln \cos x) = \frac{1}{\cos x}\times(-\sin x)$
$= -\tan x$.

(b) Area $= \int_0^1 (e^{3x} - 1 + x^2)\,dx$

$= \left[\frac{1}{3}e^{3x} - x + \frac{x^3}{3}\right]_0^1$

$= \frac{1}{3}e^3 - 1 + \frac{1}{3} - \frac{1}{3}e^0$

$= \left(\frac{1}{3}e^3 - 1\right)$ units^2

(c) (i) $P = 2r + r\theta$

Now, $\frac{1}{2}r^2\theta = 40$
$\theta = \frac{80}{r^2}$

$P = 2r + \frac{80}{r^2}\times r$
$= 2r + \frac{80}{r}$

(ii) $\frac{dP}{dr} = 2 - \frac{80}{r^2}$

At min, $2 - \frac{80}{r^2} = 0$
$2r^2 = 80$
$r^2 = 40$
$r = 2\sqrt{10}$, $\theta = \frac{80}{(2\sqrt{10})^2}$
$= 2$

$\frac{d^2P}{dr^2} = \frac{160}{r^3}$

when $r = 2\sqrt{10}$, $\frac{d^2P}{dr^2} = \frac{160}{(2\sqrt{10})^3} > 0$

There is a min value

(iii) $\frac{1}{2}r^2\theta = x$
$\theta = \frac{2x}{r^2}$

$P = 2r + \frac{2x}{r^2}\times r$
$= 2r + \frac{2x}{r}$

$\frac{dP}{d\theta} = 2 - \frac{2x}{r^2}$

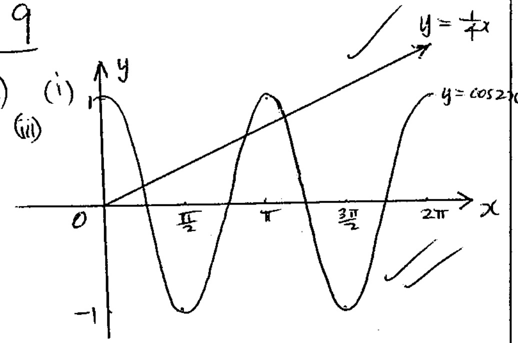
At min, $2 - \frac{2x}{r^2} = 0$
$2r^2 = 2x$
$r = \sqrt{x}$

when $r = \sqrt{x}$, $\theta = \frac{2x}{(\sqrt{x})^2}$
$= 2$

(iv) The perimeter of the sector has a minimum value when $\theta = 2$ no matter what the area.

Q9

(a) (i)
(iii)



(ii)

x	0	π	2π
y	0	0.8	1.6

(iv) $4\cos 2x - x = 0$
$\cos 2x = \frac{x}{4}$

No of solutions $= 3$
(For $0 \le x \le 2\pi$)

(b)

(i) Volume $= \int_2^5 4^2 \times \pi\,dx + \int_0^2 \pi(x^2)^2\,dx$

$= \pi\left[16x\right]_2^5 + \frac{\pi}{5}\left[x^5\right]_0^2$

$= \pi(80 - 32) + \frac{\pi}{5}\times 32$

$= 48\pi + \frac{32\pi}{5}$

$= \frac{272\pi}{5}$ units^3

(ii) Volume $= \int_0^4 \pi(5^2)\,dy - \int_0^4 \pi y\,dy$

$= \pi\int_0^4 (25 - y)\,dy$

$= \pi\left[25y - \frac{y^2}{2}\right]_0^4$

$= \pi\left[100 - \frac{16}{2} - 0\right]$

$= 92\pi$ units^3

Q10

(a) (i) Let \$M be monthly repayment

(α) $A_1 = 80000\times1.005 - M$

$A_2 = 80000\times1.005^2 - M\times1.005 - M$

(β) $A_{240} = 80000\times1.005^{240} - M\times1.005^{239} - \cdots - M$

$= 0$ after 20 years

$M(1+1.005+\cdots 1.005^{239}) = 80000\times1.005^{240}$

$$M = \frac{80000\times1.005^{240}}{\left(\frac{1.005^{240}-1}{1.005-1}\right)}$$

$\doteqdot \$573.14$

(ii) Let new monthly repayment be \$P

Amount owing after 30 repayments :

$A_{30} = 80000\times1.005^{30} - 573.14\times1.005^{29} - \cdots - 573.14$

$= 80000\times1.005^{30} - (573.14 + 573.14\times1.005 \cdots + 573.14\times1.005^{29})$

$$= 80000\times1.005^{30} - \left[\frac{573.14(1.005^{30}-1)}{1.005-1}\right]$$

$\doteqdot \$74\,411.04$

$A_{210} = 74411.04\times1.00625^{210} - P\times1.00625^{209} - P\times1.00625^{208} - \cdots - P$

After 20 years $A_{210} = 0$

$74\,411.04\times1.00625^{210}$

$= P(1+1.00625+1.00625^2+\cdots+1.00625^{209})$

$$P = \frac{74\,411.04\times1.00625^{210}}{\left(\frac{1.00625^{210}-1}{1.00625-1}\right)}$$

$\doteqdot \$637.30$

(b) (i) let $y = \sec x = (\cos x)^{-1}$

$$\frac{dy}{dx} = -(\cos x)^{-2}\times(-\sin x)$$

$$= \frac{1}{\cos^2 x}\times\sin x$$

$$= \frac{1}{\cos x}\times\frac{\sin x}{\cos x}$$

$$= \sec x\tan x$$

(ii) let $y = \sec x + \tan x$

$$\frac{dy}{dx} = \sec x\tan x + \sec^2 x$$

$$= \sec x(\tan x + \sec x)$$

(iii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x\,dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx$$

$$= \Big[\ln(\sec x+\tan x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \ln\left(\sec\tfrac{\pi}{3}+\tan\tfrac{\pi}{3}\right) - \ln\left(\sec\tfrac{\pi}{4}+\tan\tfrac{\pi}{4}\right)$$

$$= \ln(2+\sqrt{3}) - \ln(\sqrt{2}+1)$$

$$= \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right)$$

[Must be in this form]